Essential Mathematics & Statistics for Science: Second Edition by Graham Currell & Antony Dowman (John Wiley & Sons)

Answers to 'Q' Questions

5 Logarithmic & Exponential Functions

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Q5.1

i) $10^2 \times 10^3 = 10^{2+3} = 10^5$

ii) $(10^2)^3 = 10^{2\times3} = 10^6$

iii) $10^3 \div 10^2 = 10^{3-2} = 10^1 = 10$

iv) $10^3 \div 10^{-2} = 10^{3-(-2)} = 10^{3+2} = 10^5$

v) $e^2 \times e^0 = e^{2+0} = e^2$

vi) $e^3 \times e^2 = e^{3+2} = e^5$

vii) $e^5 \div e^3 = e^{5-3} = e^2$

viii) $(e^2)^3 = e^{2\times3} = e^6$

Q5.2

i) $10^{3.4} = 2511.88...$

ii) $10^{-0.45} = 0.3548..$

iii) $10^0 = 1$

iv) $e^{-4.2} = 0.657..$

v) $e^1 = 2.718...$

vi) $e^0 = 1$

Q5.3

i) log(348) = 2.541…

ii) log(34.8) = 1.541…

iii) log(3.48) = 0.541…

iv) log(100) = 2

v) log(0.01) = -2

vi) log(0.5) = -0.301…

vii) log(2) = 0.301…

viii) log(20) = 1.301…

ix) $\ln(e^1) = 1$

x) $\ln(10) = 2.302\ldots$

Q5.4

i) $e^x = 22$

$x = \ln(22) = 3.091\ldots$

ii) $e^{3x} = 12$

$3x = \ln(12) = 2.4849\ldots$

$x = 2.4849\ldots/3 = 0.828\ldots$

iii) $e^{-2x} = 3.1\times10^{-3}$

$-2x = \ln(3.1\times10^{-3}) = -5.776\ldots$

$2x = 5.776\ldots$

$x = 5.776\ldots/2 = 2.888\ldots$

iv) $\ln(x) = 1.68$

$x = e^{1.68} = 5.366..$

v) $\ln(-0.81x) = 0.37$

$-0.81x = e^{0.37} = 1.448$

$$x = \frac{1.448}{-0.81} = -1.788$$

vi) $10^x = 18$

$x = \log(18) = 1.255\ldots$

vii) $10^{2x} = 18$

$2x = \log(18) = 1.255\ldots$

$x = 1.255\ldots/2 = 0.627\ldots$

viii) $10^{-x} = 1.0 \times 10^{-7}$

$-x = \log(1.0 \times 10^{-7}) = -7$

$x = 7$

ix) $\log(x) = -9.3$

$x = 10^{-9.3} = 5.01 \times 10^{-10}$

x) $\log(-2.6x) = 3.2$

$-2.6x = 10^{3.2} = 1584.9..$

$$x = \frac{1584.9..}{-2.6} = -609.57..$$

Q5.5

i) $\log(10^{-0.3}) = -0.3$

ii) $\ln(e^{0.62}) = 0.62$

iii) $\log(2) = 0.30$ approx

iv) $\log(20) = \log(2\times10) = \log(2) + \log(10) = 0.30 + 1 = 1.3$

v) $\log(0.5) = \log(1\div2) = \log(1) - \log(2) = 0 - 0.30 = -0.30$

vi) $\ln(1000) = 2.30\times\log(1000) = 2.30 \times 3 = 6.90$

vii) $\ln(2) = 2.30 \times \log(2) = 2.30 \times 0.30 = 0.69$

viii) $\log(2^{3.1}) = 3.1\times \log(2) = 3.1 \times 0.30 = 0.93$

Q5.6

i)
$2e^{3p} := 22$ divide by 2
$e^{3p} := 22/2 = 11$ take 'ln'
$\ln(e^{3p}) = 3p = \ln(11) = 2.398$ divide by 3
$p = 2.398/3 = 0.799$

ii)
$1.2 \times 10^{2p} = 18$ divide by 1.2
$10^{2p} = 18/1.2 = 15$ take 'log'
$2p = \log(15) = 1.176..$ divide by 2
$p = 1.176../2 = 0.588..$

iii)
$4^{p} = 33$
$\log(33) = \log(4^{p}) = p \times \log(4)$
$0.602p = 1.52$
$p = 2.52$

iv)
$0.2^{p} = 0.3$ take log or ln in this case
$p \times \log(0.2) = \log(0.3)$
$p = \log(0.3)/\log(0.2) = -0.524/-0.699 = 0.748$

Q5.7

i)
$V = p.A^{k}$ take logs
$\log(V) = \log(p) + k \times \log(A)$ compare with
$y = c + m \times x$
Plot $\log(V)$ against $\log(A)$, slope $m = k$ and intercept $c = \log(p)$, so $p = 10^{c}$

ii)
$E = \sigma (T+273)^{z}$ take logs
$\log(E) = \log(\sigma) + z \times \log(T+273)$ compare with
$y = c + m \times x$
Plot $\log(E)$ against $\log(T+273)$, slope $m = z$ and intercept $c = \log(\sigma)$, so $\sigma = 10^{c}$

Q5.8

i) 5.0×10^{6}

ii) 5.0×10^{4}

Q5.9

Initial loudness = 70dB

i) Doubling power density adds 3dB to loudness. New value = 73dB

ii) Increasing by 8 (= 2^{3}) adds 3×3dB = 9dB. New value = 79dB

iii) Increasing by 100: difference = $10\log(P_1/P_2) = 10\log(100) = 10 \times 2 = 20$
New value = 70 + 20 = 90dB

iv) Halving reduces the loudness by 3dB (see (i)) New value = 67dB

v) See (ii) – subtract 9dB. New value = 61dB

vi) See (iii). Subtract 20dB. New value = 50dB.

Q5.10

i) $pH = -\log(3.4\times10^{-9}) = -(-8.468..) = 8.468..$

ii) $pH = -\log(3.4\times10^{-4}) = -(-3.468..) = 3.468..$

Q5.11

i) $-\log[H^+] = 9.2$ multiply by -1
$\log[H^+] = -9.2$ inverse 'log'
$[H^+] = 10^{-9.2} = 6.31 \times 10^{-10}$ mol L^{-1}.

ii) $-\log[H^+] = 3.2$ multiply by -1
$\log[H^+] = -3.2$ inverse 'log'
$[H^+] = 10^{-3.2} = 6.31 \times 10^{-4}$ mol L^{-1}.

Q5.12

T%	**A**
Percentage Transmittance	**Absorbance**
0%	∞ (infinity)
0.1%	3
1%	2
10%	1
50%	0.30
100%	0

Q5.13

The remaining money will fall as follows:

	End of Day n	0	1	2	3	4	5	6	7
i) halving	Money, M(£)	640	320	160	80	40	20	10	5
ii) using formula	Money, M(£)	640	320	160	80	40	20	10	5

Typical calculation for (ii): If n=4
$M = 640 \times \exp(-0.693 \times 4) = 640 \times \exp(-2.772) = 640 \times 0.0625 = 40.02$

Which is close to 40.

Q5.14

i) $g = 2$ is equivalent to a doubling

ii) $g = 1.5$ is equivalent to a 50% increase (0.5 increase)

iii) $g = 0.9$ is equivalent to a fall of 10% (0.1 reduction)

iv) $g = 0.1$ is equivalent to a fall **to** 10% (0.9 reduction)

Q5.15

i) The period of change is 2 weeks.

ii) At each stage, the number of people involved will increase by a factor of '6'.
So the gain factor for this problem, $g = 6$.

iii) Using the equation:
$N_t = N_O \times g^{t/\tau}$
At the beginning, when $t = 0$, there is just one person involved, and we write, $N_O = 1$. From above, g = 6 and $\tau = 2$
So our equation becomes:
$N_t = 1 \times 6^{t/2}$
To solve the problem we need to find the value of t such that $N_t \geq 20\times10^6$.
The first step is to solve the following equation for t.
$20 \times10^6 = 6^{t/2}$
Taking logs of both sides (see 5.1 Mathematics of e, log & ln):
$\log(20 \times10^6) = \log(6^{t/2})$
$7.30 = (t/2) \times \log(6) = (t/2) \times 0.778 = t \times 0.778/2 = 0.389\ t$
$t = 7.30 / 0.389 = 18.8$ weeks
Thus we can see that by the end of 19 weeks, at least one third of the UK population would be involved - assuming of course that the chain is not broken (which thankfully it always is!).

Q5.16

i) Every 1.2 hours the population reduces to one tenth
3.6 hours is 3 times 1.2 so the population will reduce by $0.1^3 = 0.001$ (1/1000) giving a population = $2.0\times 10^6 \times 0.001 = 2.0\times 10^3$

ii) $g = 0.1$

iii) Using the equation:
$N_t = N_O \times g^{t/\tau}$ $g = 0.1$ and $\tau = 1.2$ $N_O = 2.0\times 10^6$
$N_{2.2} = 2.0\times 10^6 \times 0.1^{2.2/1.2} = 2.0\times 10^6 \times 0.1^{1.833} = 2.0\times 10^6 \times 0.0147$
$= 2.936 \times 10^4$

Q5.17

i) 11.4 days = 3 $\times$ 3.8 days which is 3 half lives. So the activity will reduce to

$$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

10.8 /8 = 1.35 s^{-1}

ii) 10 days is not a simple multiple of the half life so we use the equation

$A_t = A_O \times e^{-0.693t/3.8}$ with the value for k (= -0.693 / $T_{1/2}$) from Table 5.1

where A is the count rate (activity) and t is in days.

When t = 10 days

$A_t = 10.8 \times e^{-0.693 \times 10/3.8} = 10.8 \times e^{-1.824} = 10.8 \times 0.161 = 1.743\ s^{-1}$

Q5.18

We will use the equation:

$N_t = N_O \exp(0.693\times t/T_G)$ see Table 5.1 with t in minutes

$1.7 \times 10^6 = 2 \times 10^4 \exp(0.693\times 180/T_G) = 2 \times 10^4 \exp(124.74/T_G)$

$(1.7 \times 10^6)/(2 \times 10^4) = \exp(124.74/T_G) = 85$

$(124.74/T_G) = \ln(85) = 4.443$

$124.74 = 4.443 \times T_G$

$T_G = 124.74/4.443 = 28$ minutes

Q5.19

Using [5.24] and the value of k in Table 5.1 we can write directly:

$N_t = N_O \exp(-0.693 \times t / T_{1/2}) \exp(-0.693 \times t / T_{1/2})$ see Example 5.16

If the activity decays to 1/10, $N_t = 0.1 \times N_O$ whatever the value of N_0

$0.1 = \exp(-0.693 \times 26 / T_{1/2})$ where the times are in hours

$\ln(0.1) = (-0.693 \times 26 / T_{1/2}) = -18.018 / T_{1/2}$

$-2.303 = -18.018 / T_{1/2}$

$T_{1/2} = -18.018/-2.303 = 7.825$ hours

An alternative method is to use the general equation

$N_t = N_O \exp(kt)$ and find firstly the value of k for this example.

So, $0.1 = \exp(26k)$

$\ln(0.1) = 26k$

$k = -2.303/26 = -0.0886$ hours^{-1}

Now find the time when $N_t = 0.5 \times N_0$ using the value of k just found

$0.5 = \exp(-0.0886\ t)$

$\ln(0.5) = -0.0886\ t$

$t = -0.693/-0.0886 = 7.82$ hours as before, and this is the half life $T_{1/2}$

Q5.20

Using [5.24] and the value of k in Table 5.1 we can write directly:

$Q_t = Q_O \exp(- t /\tau)$ where Q represents the charge

$T = CR = 0.01 \times 10^{-6} \times 330 \times 10^{3} = 3.3 \times 10^{-3}$ s
If the charge discharges to 1/100, $Q_t = 0.01 \times Q_O$ whatever the value of Q_0
$0.01 = \exp(-t / (3.3 \times 10^{-3}))$ where t is in seconds
$\ln(0.01) = -t/(3.3 \times 10^{-3})$
$-4.605 = -t / (3.3 \times 10^{-3})$
$t = 4.605 \times (3.3 \times 10^{-3}) = 0.0152$ s or 15.2 ms

Q5.21

i) We can put time $t = 0$ when $N_0 = 450$, then:
$450 = N_O\exp(k\times0) = N_O$ giving the equation:
$N_t = 450\,e^{kt} = 450\exp(kt)$
We can then substitute the values, $N_t = 620$ at time, $t = 10$.
$620 = 450\exp(k \times 10)$
Dividing both sides by 450:
$\exp(k\times10) = 620/450 = 1.378$
Taking natural logs of both sides;
$\ln(\exp(k \times 10)) = \ln(1.378) = 0.320$
$\ln(\exp(k \times 10)) = k\times10$
Hence
$k = 0.320 / 10 = 0.0320$ 'hour^{-1}'
The complete equation is therefore:
$N_t = 450\exp(0.0320\times t)$

ii) Substituting for $t = 12$ gives:
$N_{12} = 450\exp(0.0320 \times 12) = 661$

Q5.22

i) If $t = 25$ days, $N_{25} = 3500 \times \exp(0.02 \times 25) = 3500 \times \exp(0.5) = 3500 \times 1.649 = 5771$

ii) If $t = 50$ days, $N_{50} = 3500\exp(0.02 \times 50) = 3500 \times \exp(1) = 3500 \times 2.718 = 9514$

iii) If $t = 75$ days, $N_{75} = 3500\exp(0.02 \times 75) = 3500 \times \exp(1.5) = 3500 \times 4.482 = 15686$

Q5.23

$N_t = N_O\exp(kt)$ and we can use the fact that $N_1 = 1.1\ N_0$ as the population increases by 10% (= 0.1) every week. So when $t = 1$ week
$1.1 = \exp(k)$
$k = \ln(1.1) = 0.0953$ week^{-1}
The equation for $N_0 = 100$ becomes
$N_t = 100\exp(0.0953t)$

Q5.24

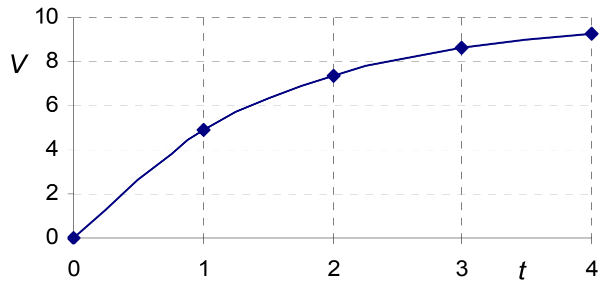
i) Using the equation:

$V = V_0\{1-\exp(-t/\tau)\}$

with $\tau = 1.5$, and $V_0 = 10$

values for V are calculated for values of t as below, and a graph drawn.

t	0	1	2	3	4
V	0.00	4.87	7.36	8.65	9.31



ii) Reading from graph for $t = \tau = 1.5$ gives $V \approx 6.5$ so $V/V_0 \approx 0.65$

$t = \tau = 3.0$ gives $V \approx 8.5$ so $V/V_0 \approx 0.85$

iii) Substituting into the equation for :

$t = \tau = 1.5$ gives $V = 6.32$ so $V/V_0 \approx 0.632$

and

$t = 2\tau = 3.0$ gives $V = 8.65$ so $V/V_0 \approx 0.865$

Q5.25

$I_d = 25.0\ e^{(-kd)}$ $\qquad$ $I_5 = 5.6$ lumens (lm) at a depth, $d = 5.0$ m

$5.6 = 25.0\ e^{(-5k)}$

$\ln(5.6/25) = \ln(0.224) = -1.496 = -5k$

$k = 1.496/5 = 0.299\ m^{-1}$

Using this value of k when $d = 10$ m

$I_{10} = 25.0\ e^{(-0.299 \times 10)} = 25.0 \times e^{-2.99} = 25.0 \times 0.050 = 1.257$ lumens

Q5.26

Using [5.17], $k = -0.012\ s^{-1}$ $\qquad$ the slope is the value of k

From Table 5.1, $k = -0.693/\ T_{1/2}$

So $T_{1/2} = -.0693/k = -0.693/ -0.012 = 57.75$ s

Q5.27

Taking natural logs of the equation $C = C_O\exp(-Kt)$ gives

$\ln(C) = \ln(C_O) - Kt$

giving:

$\ln(C) = -Kt + \ln(C_O)$

Comparing this equation with

$y = mx + c$

we can see that the slope of ln(C) plotted against t will be:

$m = -K$

and the intercept will be

i) $c = \ln(C_O)$

$K = -m$

Hence, if the slope, $m = -0.61$, then

$K = 0.61\ \text{hour}^{-1}$

ii) $c = \ln(C_O) = 4.1$

$Co = \exp(4.1) = 60.3$