7.6 Bayesian odds

7.6.1 Introduction

The basic ideas of probability have been introduced in Unit 7.3 in the book, leading to the concept of conditional probability. We now introduce the Bayesian approach to probability that uses a 'likelihood ratio' to quantify the way in which new information can change the expectation, or 'odds', that a particular condition is true. In forensic terms, a court would be interested in whether the evidence, E, is sufficiently strong to persuade a jury to accept the statement, S, of events proposed by the prosecution, or to accept the proposition from the defence that the statement is false, $\overline{S}$. We introduce the concepts of 'prior odds' and 'posterior odds' as the 'odds' before and after introducing a new piece of evidence. Bayesian statistics have particular advantages due to their simplicity in combining the cumulative 'odds' of a range of different contributing factors.

7.6.2 Conditional and Joint Probabilities

It is important to be very clear about our definitions of probabilities.

$p(A|B)$ is the **conditional probability** (7.4.7) that outcome A will occur, given that condition B already exists.

We now define **$p(A \cap B)$** as the **joint probability** that both outcomes A and B are true simultaneously.

The joint probability of A occurring with B must be the same as the joint probability of B occurring with A, and we can deduce that:

$$p(A \cap B) = p(B \cap A) \qquad [7.32]$$

Example 7.30

Using the data from example Example 7.20, calculate the *joint* probability that a man, selected at random, will *both* have the particular disease AND test positive, $p(\mathrm{P} \cap \mathrm{D})$, i.e. the probability of recording a *true positive*.

The probability that a man, selected at random, will have the disease, $p(D) = 0.05$. Given that he has the disease, the man must then test positive to satisfy the joint probability. The *conditional* probability that he will test positive, given that he has the disease, is $p(P|D) = 0.9$.

Hence the *joint* probability of having the disease AND testing positive is given by:

$$p(P \cap D) = p(P|D) \times p(D) = 0.9 \times 0.05 = 0.045$$

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We can write the general relationship:

$$p(A \cap B) = p(A|B) \times p(B) \qquad [7.33]$$

Example 7.31

In a group of 10 people, one person, *G*, was responsible for breaking a window.

The probability that *G* will have glass on his/her clothing is estimated to be $p(glass|G) = 0.99$.

The probability that a different person, $\overline{G}$, may have glass on his/her clothing is estimated to be $p(glass|\overline{G}) = 0.2$.

Calculate the probabilities that a person selected at random from the group of 10:

i) has glass on his/her clothing and is the person (*G*) who broke the window $= p(glass \cap G)$

ii) has glass on his/her clothing and is *not* the person ($\overline{G}$) who broke the window $= p(glass \cap \overline{G})$

Answer

i) Probability that the person *G* will be selected at random, $p(G) = 1/10 = 0.1$
Given that $p(glass|G) = 0.99$ we can use [7.33] to write:
$p(glass \cap G) = p(glass|G) \times p(G) = 0.99 \times 0.1 = 0.099$

ii) Probability that a person selected at random will *not* be *G*,
$p(\overline{G}) = 1 - p(G) = 1.0 - 0.1 = 0.9$
Given that $p(glass|\overline{G}) = 0.2$ we can use [7.33] to write:
$p(glass \cap \overline{G}) = p(glass|\overline{G}) \times p(\overline{G}) = 0.2 \times 0.9 = 0.18$

Q7.33

A population of 200,000 possible suspects is screened using a DNA test to match a specimen collected from a crime scene. It is estimated that the DNA results would produce a random, false positive, match for 1 person in 1.0 million. Assume that the person, *K*, who left the specimen, would definitely provide a DNA match.

Calculate the probabilities that a person selected at random from the 200,000 possible suspects:

i) has a DNA match and is the person who left the DNA specimen,

ii) has a DNA match but is *not* the person who left the DNA specimen.

7.6.3 Bayes' Rule

From [7.33] we have:

$$p(A \cap B) = p(A|B) \times p(B)$$

Similarly using [7.32] we could write:

$$p(B \cap A) = p(B|A) \times p(A)$$

and, using [7.31], we can combine the two equations above, to give:

$p(\mathrm{A|B}) \times p(\mathrm{B}) = p(\mathrm{B|A}) \times p(\mathrm{A})$

which then gives **Bayes' Rule**:

$$p(A \mid B) = p(B \mid A) \times \frac{p(A)}{p(B)} \qquad [7.34]$$

The above equation is very important because it '*transposes the conditional*'.
It changes from the probability of *B*, given that *A* is true, to the probability of *A*, given that *B* is true. This is a subtle, but very significant, change.

For example, in a diagnostic test (e.g. in Example 7.20) the scientist is primarily concerned with the *accuracy of the test*, i.e. "what is the probability that the test will give a positive result, given that the patient does have the disease?". However, *from the patient's point of view*, the more important question (with the *transposed conditional*) is "what is the probability that I have the disease, given that I have just tested positive?".

The probability, $p(B)$, of getting outcome *B* will be the sum of the probability of getting *B* with *A* true plus the probability of getting *B* with *A* false ($\overline{A}$ or not *A*):

$$p(B) = p(B \cap A) + p(B \cap \overline{A}) = p(B \mid A) \times p(A) + p(B \mid \overline{A}) \times p(\overline{A}) \qquad [7.35]$$

We can substitute $p(B)$ from [7.35] into [7.34] and obtain:

$$p(A \mid B) = \frac{p(B \mid A) \times p(A)}{\{p(B \mid A) \times p(A) + p(B \mid \overline{A}) \times p(\overline{A})\}} \qquad [7.36]$$

This is a very useful rearrangement of Bayes' Rule, as we will see in the following example.

Example 7.32
Using the data again from example Example 7.20, we know that the probability that a man with the disease, will test positive in a diagnostic test is given by, $p(P|D) = 0.9$. i.e. there is a 90% chance that a man with the disease will test positive.
But what is the probability, $p(D|P)$, that a man selected at random, who then tests positive, will actually have the disease?
(Note the change of the conditional in this problem)

To answer this we use [7.36]:

$$p(D\mid P) = \frac{p(P\mid D)\times p(D)}{\left\{p(P\mid D)\times p(D) + p\left(P\mid \overline{D}\right)\times p\left(\overline{D}\right)\right\}}$$

We have the necessary data from Example 7.15:
$p(P|D) = 0.9$, $p(P|\overline{D}) = 0.2$, $p(D) = 0.05$ and $p(\overline{D}) = 1 - p(D) = 0.95$
Combining these gives:

$$p(D\mid P) = \frac{0.9\times 0.05}{\{0.9\times 0.05 + 0.2\times 0.95\}} = 0.191$$

This is the same result as was achieved in [7.32].

Example 7.33
Using the data from example Example 7.31, calculate the probability, $p(G|glass)$, that a person, selected at random, who is found to have glass on his/her clothing, is the person, G, who broke the window

Using [7.35] and data from Example 7.31:

$$p(G\mid glass) = \frac{p(glass\mid G)\times p(G)}{\left\{p(glass\mid G)\times p(G) + p\left(glass\mid \overline{G}\right)\times p\left(\overline{G}\right)\right\}} = \frac{0.099}{0.099 + 0.18} = 0.355$$

Q7.34
Using the data from example Q7.33, calculate the probability that a person, selected at random from the 200,000, and who is found to have a DNA match, will be the person, K, who was responsible for leaving the specimen.

7.6.4 Bayesian Odds

If the possible outcomes of a statistical trial are either A or $\overline{A}$, then: $p(A) = 1 - p(\overline{A})$

The **Odds of *A***, $O(A)$, are an alternative way of expressing the likelihood of a particular event, A, being true, and are defined:

$$O(A) = \frac{p(A)}{p\left(\overline{A}\right)} = \frac{p(A)}{1 - p(A)} \qquad [7.37]$$

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This equation can be rearranged to give:

$$p(A) = \frac{O(A)}{1+O(A)} \qquad [7.38]$$

The relationships between probabilities, '*p*', and odds, '*O*', are illustrated in Table 7.3:

$p(A)$	0.01	0.1	0.5	0.9	0.99
$p(\bar{A})$	0.99	0.9	0.5	0.1	0.01
Odds, $O(A)$	0.0101	0.111	1	9	99
Odds *For*	-	-	Evens	9:1	99:1
Odds *Against*	99:1	9:1	Evens	-	-

Table 7.3 Odds and Probabilities

Example 7.34
Using the data again from example Example 7.20,
i) Calculate the odds, *before* being tested, that a man, selected at random, has the particular disease, given that $p(D) = 0.05$
ii) Use the result from Example 7.32 to calculate the *conditional* odds, $O(D|P)$, that a man who *has tested* positive does indeed have the particular disease.
iii) How *significant* is a positive test result to a man selected at random?

Answer:
i) Using [7.36], we can calculate the odds:
$O(D) = p(D) / \{1 - p(D)\} = 0.05 / 0.95 = 0.0526$
ii) From Example 7.18, $p(D|P) = 0.191$
$O(D|P) = p(D|P) / \{1- p(D|P)\} = 0.191 / 0.809 = 0.236$
iii) As far as the man is concerned, the positive result from the test has made it 0.236/0.053 = 4.5 *times more likely* that he does have the disease.

From example Example 7.20:

Odds *before* the test are called **Prior Odds** [7.39]
$O(D) = p(D) / \{1- p(D)\} = 0.0526$

Odds *after* testing positive are called **Posterior Odds** [7.40]
$O(D|P) = p(D|P) / \{1- p(D|P)\} = 0.236$

Using Bayes Rule from [7.33], we can write:

$$p(D \mid P) = p(P \mid D) \times \frac{p(D)}{p(P)}$$

$$p(\bar{D} \mid P) = p(P \mid \bar{D}) \times \frac{p(\bar{D})}{p(P)}$$

$$O(D\,|\,P) = \frac{p(D\,|\,P)}{p(\overline{D}\,|\,P)} = \frac{\left\{p(P\,|\,D)\times\dfrac{p(D)}{p(P)}\right\}}{\left\{p(P\,|\,\overline{D})\times\dfrac{p(\overline{D})}{p(P)}\right\}} = \frac{p(P\,|\,D)}{p(P\,|\,\overline{D})}\times\frac{p(D)}{p(\overline{D})}$$

which simplifies to

$$O(D\,|\,P) = \frac{p(P\,|\,D)}{p(P\,|\,\overline{D})}\times O(D) \qquad [7.41]$$

We can now define the **Likelihood Ratio** by using the equations:

Posterior Odds = Likelihood Ratio × Prior Odds [7.42]

$$\text{Likelihood Ratio, } LR = \frac{p(P\,|\,D)}{p(P\,|\,\overline{D})} \qquad [7.43]$$

The *effect of the test* has been to change the Odds of the man having the disease by multiplying the Prior Odds by the Likelihood Ratio.

7.6.5 Calculating Likelihood Ratios

The Likelihood Ratio (*LR*) can be calculated directly by using a quantitative assessment of the evidence, i.e. calculating the alternative probabilities of a given outcome, *P*, given two exclusive conditions, *D* and $\overline{D}$.

In a courtroom situation, a jury may be faced with the problem of deciding whether a statement describing a situation or an event, is *true*. The prosecution may claim that the statement is true, *S*, while the defence claims that the statement is not true, $\overline{S}$. Evidence, *E*, may be presented which changes the odds, $O(S|E)$, of the statement being true, *S*, given the evidence.

A 'new' piece of evidence, *E*, will change the odds that the statement is true. The Likelihood Ratio (*LR*) for this new piece of evidence is given by the ratio of the probabilities of the 'evidence' occurring given that the statement is either true, $p(E|S)$, or not true, $p(E\,|\,\overline{S})$:

$$\text{Likelihood Ratio, } LR = \frac{p(E\,|\,S)}{p(E\,|\,\overline{S})} \qquad [7.44]$$

Then using [7.40] we get:

$$\underset{\text{Posterior Odds}}{O(S\,|\,E)} = \underset{\text{Likelihood Ratio}}{\frac{p(E\,|\,S)}{p(E\,|\,\overline{S})}} \times \underset{\text{Prior Odds}}{O(S)} \qquad [7.45]$$

supporting statement, *given the new evidence*	due to new evidence	supporting statement

An important aspect of the above equation, [7.44], for the forensic scientist, is the fact that the data required to calculate the Likelihood Ratio is in a form that is produced by scientific investigation. It is necessary only to calculate the probabilities of observing the evidence, E, given the two conditions that the statement is either true or not true.

The equation then provides the posterior odds, $O(S|E)$, with the *transposed conditional*, i.e. the probability of the statement being true, given the new evidence. The posterior odds are presented in a form appropriate for consideration in court.

Example 7.35
Use the data from Example 7.20 to calculate the Likelihood Ratio for having the disease, given a positive result from the diagnostic test.

From Example 7.20, $p(P|D) = 0.9$ and $p(P|\overline{D}) = 0.2$

Using [7.42]

Likelihood Ratio, $LR = \dfrac{p(P\,|\,D)}{p(P\,|\,\overline{D})} = \dfrac{0.9}{0.2} = 4.5$

This agrees with the increased odds calculated in Example 7.20.

Example 7.36
Assuming that the proportion of left-handed people is 10%, and that it is known that the person, G, who committed a crime was *definitely* left-handed, calculate the likelihood ratio for the evidence of left-handedness.

The probability of being left-handed for the culprit, $p(L|G) = 1.0$
For a randomly selected person, the probability of being left-handed, $p(L|\overline{G}) = 1/10 = 0.1$
Hence, using [7.43] the likelihood ratio for the evidence of left-handedness becomes:

$$LR = \frac{p(L\,|\,G)}{p(L\,|\,\overline{G})} = \frac{1.0}{0.1} = 10$$

Q7.35
For the problem outlined in Q7.34 and Q7.33, calculate

i) Prior Odds that a randomly selected person is the person, K, who left the sample, $O(K)$,
ii) Likelihood Ratio due to the evidence, E, of a DNA match,
iii) Posterior Odds that a randomly selected person, who gives a DNA match, is the person, K, who left the sample, $O(K|E)$,
iv) Does the result in (iii) agree with the result in Q7.34?

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Q7.36
A particular blood group is present in only 5% of the population.
If it is known that the person who committed a particular crime had this blood group, calculate the likelihood ratio for this evidence when used to assess whether a suspect committed the crime. Assume that the experimental measurement of the blood group is 100% accurate.

The advantage of using the Bayesian, 'Odds', approach is that it is easy to combine different factors that lead to an overall probability, by simply multiplying the relevant Likelihood Ratios. However, for combination by simple multiplication, it is necessary that the probabilities of the two tests are *independent* of each other.

Example 7.37
In example Example 7.31, a witness states that the person, *G*, who broke the window was left-handed. Assuming that the proportion of left-handed people is 10%, calculate the posterior odds that *G* is the person who has been randomly selected from the group of 10, and who is both left-handed *and* has glass on his clothing.

Probability of selecting the person, *G*, at random from a group of 10, $p(G) = 0.1$.
Hence the prior odds for randomly selecting, *G*, before considering the evidence:

$$O(G) = \frac{p(G)}{1 - p(G)} = \frac{0.1}{1.0 - 0.1} = \frac{0.1}{0.9} \approx 0.11$$

Using data from Example 7.31, the Likelihood Ratio for discovery of glass on clothing,

$$LR(glass) = \frac{p(glass\,|\,G)}{p(glass\,|\,\overline{G})} = \frac{0.99}{0.2} = 4.95 \approx 5$$

From Example 7.22, the Likelihood Ratio for 'left-handedness',

$$LR(left) = \frac{p(left\,|\,G)}{p(left\,|\,\overline{G})} = \frac{1.0}{0.1} = 10$$

The posterior odds, taking both tests into account, (we assume that the probability of glass on the clothing is independent of whether the culprit is left-handed or not)

$$O(G|\text{glass,left}) = LR(glass) \times LR(left) \times O(G) = 5 \times 10 \times 0.11 = 5.5$$

It is not possible to use simple multiplication of likelihood ratios if the two pieces of evidence are not independent. For example, if the pieces of evidence were that a person was *believed* to be male and that he (or she) was over 6 feet tall, then the likelihood ratio for the height will be different depending on whether the person is actually male or female. In these situations it is necessary to investigate the probabilities of the different options using a form of 'probability tree'. Such calculations are very dependent on the particular situation involved and are not covered in this book.